Instability of pole singularities for the Chazy equation

Instability of pole singularities for the Chazy equation

Satyanad Kichenassamy†

School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church Street S E, Minneapolis, MN 55455-0487, USA

Abstract. We prove that the negative resonances of the Chazy equation (in the sense of Painlevé analysis) can be related directly to its group-invariance properties. These resonances indicate in this case the instability of pole singularities. Depending on the value of a parameter in the equation, an unstable isolated pole may turn into the familiar natural boundary, or split into several isolated singularities. In the first case, a convergent series representation involving exponentially small corrections can be given. This reconciles several earlier approaches to the interpretation of negative resonances. On the other hand, we also prove that pole singularities with the maximum number of positive resonances are stable. The proofs rely on general properties of nonlinear Fuchsian equations.

1. Introduction

The method of pole expansions is a versatile tool for the investigation of singularities of ordinary differential equations (ODEs) and partial differential equations (PDEs). Going as far back as the work of Briot and Bouquet, and Kowalewska, this method acquired new significance when it was found that symmetry reductions of equations integrable by inverse scattering usually have the Painlevé property in the strong sense: all of their movable singularities are described by the method of pole expansions [3, 4]. Conversely the inverse scattering transform suggested new approaches to the study of the Painlevé equations themselves.

We are interested in the case of the Chazy equation

$$y''' - 2yy'' + 3y'^2 = 0 \qquad (1)$$

where this method apparently fails to provide information on the general solution of the equation under consideration, because the pole expansion has fewer free parameters than the order of the equation, and therefore does not seem to represent the general solution. The simplest issue is this: the Chazy equation has order three and possesses the exact solution

$$-6/x.$$

However, this is, up to translations, the only solution with a simple pole. The 'pole expansion' reduces in this case to its first term. This solution should be embedded in a three-parameter family of solutions: the fact that there is only a one-parameter family of solutions with simple poles shows that simple poles are *unstable under perturbation,* but does not give further information. (More background information on the Chazy equation

† Current address: Max-Planck-Institut für Mathematik in den Naturwissenschaften, Inselstraße 22-26, D-04103 Leipzig, Germany. E-mail address: kichenas@mis.mpg.de

is given in section 1.3.) Kruskal *et al* (see [18, 27]) suggested that the failure of the pole expansion was due to the omission of exponentially small terms in the expansion of the general solution $y(x)$. Furthermore, this representation should not be valid in a full neighbourhood of $x = 0$ in general, but only in a sector, as in the case of expansions near an irregular singular point. It turns out in this case that the expansion in powers of exponentials is convergent, which is not expected at an irregular singularity. On the other hand, the perturbative approach [14, 17] shows that there is a formal series solution $y(x; \varepsilon) = y(x) + \varepsilon y_1(x) + \cdots$, which does contain the full number of arbitrary parameters. Also, the perturbative approach accounts for the apparent paradox that the solutions of the linearized equation have solutions which are more singular than any actual solution of the equation: they are related to the variation of free parameters in the general solution, at a fixed location away from the singularity. However, all of the terms in the expansion in powers of ε have only power singularities, and do not contain exponential terms. There is also no indication that the actual solution cannot be continued around the singularity.

Another difficulty comes from the general 'class XII' equation of Chazy:

$$y''' - 2yy'' + 3y'^2 = E(6y' - y^2)^2 \tag{2}$$

where $E = 4/(36 - k^2)$, with $k \geqslant 0$, omitting the 'complementary terms'. This equation has the same special solution, but the general solution can be completely different, and is in fact rational for some values of k. It is possible for the simple pole to split into two or more poles by perturbation. This effect cannot be captured adequately in a series representation in powers of x: take a solution with two poles at $x = 0$ and $x = \alpha$. The radius of convergence of a pole expansion around $x = 0$ cannot be greater than $|\alpha|$, and therefore tends to zero if there is a confluence of the two singularities ($\alpha \to 0$). The perturbative approach can correctly describe what happens at a fixed location away from the singularities, but its asymptotics as $x \to 0$ do not describe the singularity correctly. This can be seen in the treatment of the first example of [14, section 5] where the confluence can only be ascertained by explicitly summing the pole expansion.

We address these difficulties as follows.

(i) In considering the perturbative solution $y(x; \varepsilon)$, it is not permissible to exchange the limits $x \to 0$ and $\varepsilon \to 0$. This explains why no exponential terms are generated there.

(ii) To recover the results of the perturbative approach from the exponential expansion, it is necessary to vary the free parameters in a very particular way, which we relate to the group invariance of the Chazy equation.

(iii) It is possible to describe confluence phenomena analytically by means of a Cole–Hopf transformation: if $u(x; \varepsilon)$ is an analytic family of functions where a pair of simple zeros coalesces at $x = 0$, the family u'/u has a confluence of poles. Note that we can allow u to be analytic in both arguments, whereas the pole expansion of u'/u would have a vanishingly small radius of convergence, as explained above.

It is convenient to describe all of the above issues as relating to the stability of singular behaviour, because of the close similarity to the stability of solitary waves in nonlinear wave equations. Note that some authors (in particular, Bureau) define a 'stable equation' as one which possesses the Painlevé property. We are interested in a different issue, namely whether the leading asymptotics of a particular solution of an ODE or PDE are stable under perturbation of the solution. Consider, for instance, real-valued solutions of the equation

$$u_{tt} - u_{xx} - u_{yy} = e^u$$

which has the special solution $e^u = 2/t^2$; then [22] all nearby solutions are such that e^u has an expansion in powers of T and $T \ln T$, where $T = t - \psi(x, y)$, with ψ small, and the

first term is $\ln(2/T^2)$. It is not possible to exclude logarithms in general, but the leading order of the singularity remains the same. The singularity of the solution $\ln(2/t^2)$ would be termed 'stable' in the sense of the present paper.

The main technical tool in the proofs of our results is the Fuchsian algorithm ([20, 24] and earlier references therein). The basic step is to show that the method of pole expansions for ODEs or PDEs amounts to seeking solutions of a nonlinear ordinary or partial differential equation with a regular singularity, which may include logarithmic terms. As far as the present discussion is concerned, note that several authors (in particular, Bureau, see also Conte *et al* [14]) have identified ARS–WTC resonances with the indices of a linear Fuchsian ODE, and have observed in many cases that the linearization of the equation itself is Fuchsian. On the other hand, we have shown, see for example [24], that under very general circumstances the equation itself can be reduced to a nonlinear Fuchsian ODE or PDE, without linearizing. We have also shown that the initial-value problem for a Fuchsian PDE can be solved in both the analytic and non-analytic cases. The Chazy equation itself can be reduced to Fuchsian form in several different ways, see the proofs of theorems 7 and 8, later. The Fuchsian algorithm is not restricted to 'integrable' problems, but is in fact useful in analysing singularity formation in more general nonlinear wave equations, see [20, 22, 23]. Of course, in the linear case, none of the above issues arises: if a linear equation has solutions with leading order x^{ν_1}, x^{ν_2}, ..., the general solution is simply a linear combination of these, and its singular behaviour is apparent. Confluence of singularities of linear Fuchsian equations involves varying the coefficients of the equation, rather than perturbing one solution of a fixed equation.

1.1. The method of pole expansions

The principle of the method is familiar: given an equation, substitute for the unknown u a series of the form

$$u = x^\nu \sum_{j \geqslant 0} u_j x^j \tag{3}$$

and identify like powers of x. This determines first ν and u_0, hence the *leading balance* $u \sim u_0 x^\nu$. The other coefficients are usually determined by a recurrence relation of the form

$$Q(j)u_j = F_j(u_0, \ldots, u_{j-1}). \tag{4}$$

The zeros of Q are called *resonances*.

When $Q(j) = 0$ and F_j also vanishes, the coefficient u_j is arbitrary. If $Q(j) = 0$ but F_j does not vanish, the series (3) must be corrected by the addition of logarithmic terms, and one can predict their form rather precisely [24]. The requirement that no logarithmic terms are needed sets rather strong constraints on the equation; this observation was one of the ingredients of the 'Painlevé test' in its original form, because many symmetry reductions of integrable equations do have this property. We do not discuss the modern status of this test, referring the reader to recent reviews [2, 21, 26–28] and their references. The method can be extended to partial differential equations, in which case it is known as the WTC method [24, 35].

Therefore, as far as this paper is concerned, the upshot is that singularities of the form (3) are expected to be stable under perturbations, in the sense defined in section 1.2, provided that the series (3) has the maximum number of arbitrary constants in it: any nearby solution must have a singularity of the same type, with a possible shift in location.

1.2. Stability of singular behaviour

The stability of a singular solution will be defined by analogy with the case of orbital stability of solitary waves in translation-invariant problems: a solitary wave u is (orbitally) stable if any perturbation of the initial condition for a solitary wave generates a solution which remains close to the orbit of u under translations. Thus, in the case of the Korteweg–de Vries (KdV) equation, an initial condition close to a one-soliton leads to a solution which remains close to the set of all translates of this soliton (see Strauss [32, 33], Bona *et al* [6] for the KdV case, and their references; further results for KdV-like equations are also found in [29]).

We encounter a comparable situation for movable singularities of differential equations: if an equation is translation-invariant, we expect that a small perturbation of any solution, in a domain away from the singularity, will in general result in a *shift in the singularity location*. This should be distinguished from a possible change in the type of singularity.

As an example with a simple closed-form solution, consider the equation

$$\mathrm{d}u/\mathrm{d}x = 2xu^2$$

and let us focus on real solutions to fix ideas; a similar discussion could be made for complex solutions as well. The general solution is

$$u(x) = \frac{1}{c - x^2}$$

where c is a real constant, and we consider small values of c. If c is positive, we have two stable single poles: a slight change in the value of, say, $u(1) = 1/(c - 1)$, results in a small change in the value of c, and hence in a small shift of the pole. However, if $c = 0$, we obtain an unstable double pole for $x = 0$, because a small change in the value of $u(1)$ causes the pole either to disappear or to break up into two simple poles, depending on whether c becomes negative or positive.

In the Chazy case, we will be interested in the stability of exact solutions of the form a/x in some fixed disk D around the origin. Fix also some non-zero value ξ in D. We will say that the singularity is stable if *any* solution with Cauchy data at ξ close to those of a/x has an expansion

$$a/(x - x_0) + \sum_{j \geqslant 0} u_j (x - x_0)^j$$

valid in D. We could allow logarithmic terms in the series, but they will not be needed. We, therefore, require a simple pole to perturb to another simple pole. An example of an unstable situation would be the case in which a small change in the data at ξ causes the pole to turn into a natural boundary, or to split into several poles.

1.3. The Chazy equation

We recall here some background information on equation (1). The Chazy equation came up in the course of Chazy's extension of Painlevé's program to third-order equations [10–12]. It is actually one of the 'class XII' equations (2). It is closely related to a system considered by Halphen, and its general solution can be parametrized using the solutions of a hypergeometric equation if $k > 0$, and the Airy equation if $k = 0$ (see Clarkson and Olver [13], Ablowitz and Clarkson [2], and their references).

The modern interest in this equation comes from the fact that it arises as a reduction of the self-dual Yang–Mills equations with an infinite-dimensional gauge [1, 2, 9, 34]. It arises

in connection with one of the special reductions of Einstein's equations in a Bianchi IX spacetime. It can be given a commutator representation. A particular solution is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\ln\Delta(x)$$

where $\Delta(x)$ is the discriminant modular form (see Takhtajan [34], Bureau [7, 8] and Chazy [12] for the relation to modular forms, and Koblitz [25] for background material on modular forms). This solution has the real axis as a natural boundary. It generates a three-parameter family of solutions through an SL(2) action which maps solutions to solutions: more precisely, if $y(x)$ is a solution, so is

$$\frac{ad-bc}{(cx+d)^2}y\left(\frac{ax+b}{cx+d}\right)-\frac{6c}{cx+d} \tag{5}$$

for any choice of the complex parameters a, b, c and d, subject to $ad-bc\neq 0$. There are effectively only three parameters, since scaling the parameters by a common factor does not generate a new solution. The transformed solution has, in general, a circular natural boundary. The relation of the Chazy equation to the Schwarzian derivative and invariants of SL(2) actions was elucidated by Clarkson and Olver [13], who showed how to use the group action to obtain the general solution, despite the fact that the symmetry group is not solvable.

On the other hand, the method of pole expansions generates exact solutions which have no natural boundary at all, namely

$$-\frac{6}{x-x_0}+\frac{A}{(x-x_0)^2}$$

where A and x_0 are arbitrary. This is a two-parameter deformation of the solution $-6/x$.

The problem is to decide the stability of the solution $-6/x$.

A similar problem arises for the general class XII equation, for which we have the solutions [12, 31]

$$\frac{k-6}{2(x-a)}-\frac{k+6}{2(x-b)}$$

where a and b are arbitrary. One recovers the solution $-6/x$ by confluence of a and b. The resonances corresponding to the singularities a and b are $(-1,1,k)$ and $(-k,-1,1)$, respectively.

1.4. Earlier approaches

1.4.1. Linearization. The method of Fordy and Pickering [17] considers perturbations of the solution $-6/x$ of the form

$$-\frac{6}{x}+\varepsilon u_1(x)+\varepsilon^2 u_2(x)+\cdots \tag{6}$$

and shows that one can compute the u_k recursively by solving linear equations involving the linearization of the Chazy equation at the reference solution $-6/x$. This linearization reads

$$(D+2)(D+3)(D+4)v=0$$

where $D=x\,\mathrm{d}/\mathrm{d}x$, and the corrections are sums of increasingly high powers of $1/x$ and x.

The advantage of this method is that the new series does contain three arbitrary parameters, which can be identified with the three parameters required to describe the general solution of the linearized equation.

The solutions of the linearization are more singular than $-6/x$, but this does not mean that the equation itself has more singular solutions: for example, we have

$$-\frac{6}{x-\varepsilon} = -\frac{6}{x}(1 + \varepsilon/x + \varepsilon^2/x^2 + \cdots).$$

Since the series converges only for $|x| > \varepsilon$, the series does not provide information on the behaviour of this solution as x approaches zero. In fact, $x = 0$ is a regular point for $\varepsilon \neq 0$. Similarly, the series (6) is not expected to converge for x and ε small without any further restriction, since this would predict that pole singularities perturb to Laurent expansions defined in a full neighbourhood of $x = 0$, and such expansions do not allow for the formation of a small natural boundary near the origin.

1.4.2. Exponentially small corrections.

Kruskal considered the general solution as a perturbation of the solution $-6/x + A/x^2$. In this case, the linearized equation reads

$$(x(D+4) - 2A)(D+2)(D+3)v = 0$$

where again $D = x\,d/dx$, and has therefore an *exponential solution* which involves $\exp(-2A/x)$. The appearance of non-Fuchsian terms was to be expected from the fact that the leading balance, for a second-order pole, does not involve the top-order derivatives. He derived systematically a representation of the general solution in the form

$$-\frac{6}{x} + \frac{A}{x^2}(1 + k\exp(-2A/x) + \cdots)$$

which, after including a parameter for translations, contains three parameters, but is defined at best in a sector in which the exponential is small. This sector has maximal angle π, and this restriction corresponds to the fact that the solution cannot be continued around the singularity at $x = 0$ in general, but has a natural boundary which is a circle (or a line) through $x = 0$.

This representation can be checked directly in this special case using the general solution [2]. One can see by inspection that, as A approaches zero, the natural boundary is a small circle which shrinks to a point.

However, it is not clear how to account for resonance -3 in this context, even though the solution thus obtained does contain three arbitrary parameters. Indeed, if we differentiate the expansion with respect to A or k and set $A = 0$, we cannot generate a function with a fourth-order pole at the origin, as was possible in the linearization approach. As we will see, it is possible to recover the fourth-order pole by using a *simultaneous variation* of all three parameters.

1.4.3. Remarks.

Two other approaches to the interpretation of missing parameters in pole expansions are as follows.

(1) The fact that some equations have two Painlevé series with leading orders of the form a/x and na/x where n is an integer suggests that the second series results from the confluence of n singularities of the first type (see Adler and van Moerbeke [5], Ercolani and Siggia [15], Flaschka *et al* [16], among others). However, as noted previously, it has remained difficult to obtain an analytical representation of confluence, because the pole expansion near one of the singularities has a shrinking radius of convergence.

(2) It is possible to seek solutions in powers of $s = 1/x$ rather than x. In this case, negative resonances become positive resonances of the equation in the s variable. Such a series already appears in Chazy [12]. However, this solution is only defined for x large, and there may be one or more natural boundaries which separate the domain of existence

of this solution from the origin. Also, this interpretation is not applicable if there are both positive and negative resonances.

1.5. Results of this paper

The necessary reconciliation of the linearization and exponential approaches, for equations (1) and (2) for $k = 2, 3, 4$ and 5, is obtained through the following results.

(a) Theorems 1 and 9. Pole expansions with the maximum number of arbitrary parameters represent stable singularities. This applies to the solution $(k - 6)/2x$.

(b) Theorems 2–4. Any equation with the group invariance property of the equation (5) has a one-parameter family of solutions $y(x; \varepsilon)$ such that

$$y(x; 0) = -\frac{6}{x} \qquad \frac{\partial y}{\partial \varepsilon}(x; 0) = \text{constant} \times x^{-1+r}$$

for each negative resonance r (i.e. for $r = -1, -2$ or -3). The perturbative expansion follows. This argument applies to both (1) and (2).

(c) Theorem 5. Conversely, if an equation has solution $-6/x$, it cannot have the SL(2) action (5) if the resonances do not include $-1, -2$ and -3. The existence of a full expansion of $y(x; \varepsilon)$ also follows from our proof.

(d) Theorem 6. If there is a solution with branching involving x^k, the group invariance proves the existence of pole-like singularities with resonances -1 and $-k$.

(e) Theorem 7. In the case of (1), as ε increases from zero, the isolated pole is unstable and becomes a circular natural boundary of small radius. However, the general solution is still described in terms of a convergent series of exponentials; this follows from the fact that even though the linearization of the equation can be non-Fuchsian, there still is a reduction to a nonlinear Fuchsian equation.

(f) Theorem 8. In the case of (2), for $k = 2, 3, 4$ and 5, the isolated pole does not turn into a natural boundary, but rather splits into a finite number of poles. However, the confluence pattern is restricted: all poles except one at most must coalesce. If all confluence patterns had been allowed, the resonances of the solution $-(k+6)/2x$ would have included all the negative integers from $-k$ to -1. The confluence is described analytically by representing the solution in terms of u'/u, where u is analytic in a fixed domain which includes all coalescing singularities.

2. Nonlinear Fuchsian equations

We collect here a few results on Fuchsian equations which are used later. We also include a proof of the stability of polar singularities with the maximum number of arbitrary coefficients in their expansion (3), which is similar in spirit to, but considerably simpler than, the result of [22], because of the fact that we are dealing here with an ODE rather than a PDE.

2.1. Existence results for a Fuchsian ODE

A nonlinear Fuchsian equation has the general form

$$P(D)u(x) = x F[x, u, Du, \ldots, D^{m-1}u] \tag{7}$$

where $D = x \, \mathrm{d}/\mathrm{d}x$, and P is a polynomial of degree m. There are similar definitions and results for systems, but they will not be needed here.

If the zeros of P all have negative real parts, there is precisely one solution of this equation which vanishes for $x = 0$, and it is given by a convergent series in powers of x

near $x = 0$. The proof is by iteration in a space of analytic functions, and is in this sense constructive. In particular, it could in principle be used to generate the coefficients of the expansion of the solution, but it is in practice quite cumbersome to do so.

The convergence of WTC expansions follows from the fact (see [21]) that it is possible to allow u to depend on additional 'transverse' variables, provided that spatial derivatives terms are multiplied by appropriate powers of x, see [20, 24]. This condition is satisfied in a remarkably large number of cases.

We will meet in section 3 an equation of the form

$$P(D)Du = xF \tag{8}$$

where P and F are as above. To reduce this situation to the the case of (7), let us write $u = a + xv(x)$, where a is arbitrary. One can write

$$xF = xF[0, a, 0, \ldots] + x^2 G[x, v, Dv, \ldots, D^{m-1}v]$$

for some function G. The equation now becomes, after division by x,

$$Q(D)v := P(D + 1)(D + 1)v = F[0, a, 0, \ldots] + xG$$

where the zeros of Q all have negative real parts. We now replace v by $v - b$, for a suitable constant b, to annihilate the first term on the right-hand side. In this way, we reduce the problem to an equation of the form (7), and we conclude that there is a unique solution v which vanishes at the origin. Therefore, for any a, there is a unique analytic solution of (8) which satisfies $u(0) = a$.

2.2. Stability and parameter dependence

We now prove a stability result for pole singularities of an equation with the maximum number of coefficients in their pole expansions. This result makes rigorous the intuitive argument to the effect that a series which contains as many free parameters as there are Cauchy data must represent the general solution locally.

To prove the result, one must show that these parameters are not redundant. We achieve this by a reduction to the implicit function theorem. An example of a redundant parametrization is the two-parameter family of series:

$$u(x; \varepsilon, \eta) = \sum_{j \geqslant 0} \frac{\eta^j}{(x - \varepsilon)^{j+1}}. \tag{9}$$

The parameters ε and η are redundant, because $u(x; \varepsilon, \eta) = 1/(x - \varepsilon - \eta)$: the pairs (ε, η) with the same value of $\varepsilon + \eta$ all define the same function.

To fix ideas, let us consider an autonomous equation of the form

$$(d/dx)^m u = f(u, \ldots, u^{(m-1)}) \tag{10}$$

where f is, say, a polynomial.

Let $u(x - x_0, c_1, \ldots, c_{m-1})$ be a family of solutions, which depends analytically on $(x_0, c_1, \ldots, c_{m-1})$ for $|x_0|$ and $|c_k| < a$ and $0 < |x - x_0| < b$, for some positive a and b. We have the following result.

Theorem 1. Assume that $\partial u / \partial x_0$, $\partial u / \partial c_1, \ldots, \partial u / \partial c_{m-1}$ form a linearly independent set of solutions of the linearization of (10). Then $u(x - x_0, c_1, \ldots, c_{m-1})$ is a local representation of the general solution. In particular, the assumption holds for any pole expansion with the maximum number of parameters if $v \neq 0$.

Remarks. (1) If $\nu = 0$, we are in the case of the Cauchy problem, and the series for the solution contains $m + 1$ parameters, namely the location of the initial point and the m Cauchy data. These data are clearly redundant.

(2) In the case of (9), the representation is redundant because $\partial u / \partial \varepsilon = \partial u / \partial \eta$.

Proof. Consider the reference solution $U = u(x, 0, \ldots, 0)$ for definiteness. Given any point x_1 with $0 < |x_1| < b$, we consider

$$\varphi : (x_0, c_1, \ldots, c_{m-1}) \mapsto (u(x_1), u'(x_1), \ldots, u^{(m-1)}(x_1))$$

where $u(x_1) = u(x_1 - x_0, c_1, \ldots, c_{m-1})$, and similarly for the derivatives of u. Applying the inverse function theorem to this map near $(x_0, 0, \ldots, 0)$, we conclude that any set of Cauchy data close to the data of U at x_1 coincides with the Cauchy data of a member of our family; this proves the first statement.

To prove that the linear independence condition holds in the context of the theorem, it suffices to consider the family $u(x - x_0, c_1, \ldots, c_{m-1})$, where the c_l's are the arbitrary coefficients in the expansion of u. The functions $\partial u / \partial x_0$, $\partial u / \partial c_l$ are derivatives of families of solutions, and are therefore themselves solutions of the linearized equation. It is easy to see that these derivatives all have different leading behaviour at $x = x_0$, and are therefore linearly independent. □

3. The transformation formula and negative resonances

3.1. Statement of results

Consider any equation which admits the transformation formula (5). Assume that any uniform limit of analytic solutions is also a solution. This assumption is clear for ODEs; it will allow us to extend (5) to some cases when the transformation $(ax + b)/(cx + d)$ is non-invertible, by viewing it as a limit of invertible transformations.

Let $y(x)$ be any solution. We wish to prove that there are families of solutions $y(x; \varepsilon, r)$ such that

$$y(x; 0, r) = -6/x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}\varepsilon}(x; 0, r) = x^{-1+r}$$

for $r = -1, -2$ and -3. This will account for the three 'negative resonances'. It will be apparent from the proofs that $y(x; \varepsilon; r)$ can in fact be expanded to higher order, and that the coefficients of the higher-order terms are increasingly more singular in x.

We find that the construction is possible provided that it is possible to prescribe y, y', y'' arbitrarily at one point. This construction precisely fails for the non-generic solutions $-6/x + A/x^2$. The fact that the resonance structure can be derived on the sole basis of the transformation formula implies conversely that, if we have an equation with a different resonance structure, it *cannot* admit the transformation formula (5).

Fix a solution $y(x)$ which is analytic near $x = 0$. Consider the family

$$y(x; \varepsilon) = -\frac{6}{x - \eta} + \frac{\mu}{(x - \eta)^2} y \left(\frac{-\mu}{x - \eta} \right) \tag{11}$$

where η and μ depend on ε, and are assumed to be small as $\varepsilon \to 0$. This is a special case of the transformation (5).

Our results are as follows.

Theorem 2. If $\mu y(0) - 6\eta = \varepsilon$, and μ and η are proportional to ε,

$$y(x; 0) = -6/x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}\varepsilon}(x; 0) = 1/x^2.$$

Theorem 3. Assume $\mu y(0) - 6\eta = 0$, but $6y'(0) - y(0)^2 \neq 0$. Then, if μ and η are both proportional to $\varepsilon^{1/2}$, we have

$$y(x; 0) = -6/x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}\varepsilon}(x; 0) = c/x^3$$

where $c \neq 0$.

Theorem 4. Assume $\mu y(0) - 6\eta = 0$ and $6y'(0) - y(0)^2 = 0$, but $y'' - yy' + y^3/9 \neq 0$ when $x = 0$. Then, if μ and η are both proportional to $\varepsilon^{1/3}$, we have

$$y(x; 0) = -6/x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}\varepsilon}(x; 0) = c/x^4$$

where $c \neq 0$.

Theorem 5. If an equation of order three or higher, to which the Cauchy existence theorem applies, admits the special solution $y = -6/x$, and if the linearization of the equation at this solution does not have $1/x^2$, $1/x^3$ and $1/x^4$ among its solutions, then the given equation cannot admit the SL(2) action (5).

The restriction that the solution $y(x)$ be analytic is essential. In fact, we can obtain quite different results if y admits branching.

Theorem 6. If there is a solution of the form $y(x) = x^{-1}h(x^k)$, where h is analytic, $k > 0$, $h'(0) \neq 0$ and $h(0) = (k - 6)/2$, there exist two families of solutions, $y_1(x; \varepsilon)$ and $y_2(x; \varepsilon)$, such that

$$y_1(x; 0) = y_2(x; 0) = -\frac{k + 6}{2x}$$

and

$$\frac{\mathrm{d}y_1}{\mathrm{d}\varepsilon}(x; 0) = c/x^2 \qquad \text{and} \qquad \frac{\mathrm{d}y_2}{\mathrm{d}\varepsilon}(x; 0) = c/x^{k+1}.$$

3.2. Remarks

(1) If we take $y = -6/(x - x_0)$, theorems 3 and 4 do not apply.

(2) If we take $y = -6/(x - x_0) + A/(x - x_0)^2$, theorem 3 applies, but theorem 4 does not. Indeed, in this case,

$$y' - y^2/6 = -A^2(x - x_0)^{-4}/6 \tag{12}$$

and

$$y'' - yy' + y^3/9 = (\mathrm{d}/\mathrm{d}x - (2/3)y)(y' - y^2/6) = A^3(x - x_0)^{-6}/9. \tag{13}$$

It is, therefore, not possible to make (12) vanish without having $A = 0$—in which case (13) vanishes as well. One can rephrase the assumption in theorem 4 by saying that we require $y' = y^2/6$ but $y'' \neq y^3/18$, for $x = 0$.

(3) Theorems 2–4 all apply, for example, when there is a solution for every choice of $y(0)$, $y'(0)$ and $y''(0)$. The result, therefore, holds for any third-order autonomous equation,

and hence for both equations (1) and (2) which, as we show in section 4, have completely different singularity structures.

(4) It follows from theorem 8 that theorem 6 applies to equations (2). A result similar to theorem 5 could of course be stated for this situation.

(5) An example to illustrate theorem 5 is the equation

$$y''' = 2yy'' - 3y'^2 + cy'(6y' - y^2)$$

which has the solution $-6/x$, but where the resonance equation is $(r+1)[(r+2)(r+3) - 36c] = 0$. -2 and -3 are therefore both resonances only if $c = 0$. We conclude, without computing the symmetry group of the equation, that this equation does not admit the transformation law (5) if $c \neq 0$.

3.3. Proofs

Let us begin with a computation which is used in the proofs of the first three theorems. Any solution $y(x)$ generates the one-parameter family of solutions

$$y(x; \varepsilon) = -\frac{6}{x - \eta(\varepsilon)} + \frac{\mu(\varepsilon)}{(x - \eta(\varepsilon))^2} y\left(-\frac{\mu(\varepsilon)}{x - \eta(\varepsilon)}\right).$$

If x is fixed and non-zero, and if μ and η are small as $\varepsilon \to 0$, we can expand this solution in the form

$$y(x; \varepsilon) = \frac{-6}{x} + \frac{\mu y - 6\eta}{x^2} + \frac{\mu(2\eta y - \mu y') - 6\eta^2}{x^3}$$
$$+ x^{-4}\left[-6\eta^3 + \mu\left(3\eta^2 y - 3\eta\mu y' + \frac{\mu^2}{2}y''\right)\right] + O(\eta^4, \eta^3\mu, \eta^2\mu^2, \eta\mu^3, \mu^4)$$

where $y, y', \ldots$ stand for $y(0), y'(0), \ldots$.

Any such family has the property that $y(x; 0) = -6/x$. Furthermore, it is clear that the above expansion could be pushed to all orders, and that the coefficients of the higher-order terms contain higher and higher powers of $1/x$.

Proof of theorem 2. If we take μ and η proportional to ε, in such a way that $\mu y(0) - 6\eta \sim \varepsilon$, we have $\partial y/\partial\varepsilon = 1/x^2$ for $\varepsilon = 0$. □

Proof of theorem 3. If we take μ and η proportional to $\varepsilon^{1/2}$, in such a way that $\mu y(0) - 6\eta = 0$, and if y is such that $6y'(0) \neq y(0)^2$, we have $\partial y/\partial\varepsilon = \text{constant}/x^3$ for $\varepsilon = 0$. □

Proof of theorem 4. If we take μ and η proportional to ε, in such a way that $\mu y(0) - 6\eta = 0$, and assume that $6y'(0) - y(0)^2 = 0$, but $y'' - yy' + y^3/9 \neq 0$ for $x = 0$, we find that $\partial y/\partial\varepsilon = \text{constant}/x^4$ for $\varepsilon = 0$. □

Proof of theorem 5. Consider an equation $F[u] = 0$ of order three or higher with such a group action. Solving the Cauchy problem, we can construct solutions to which each of theorems 1–3 apply. Consequently, there are differentiable families of solutions $y(x; \varepsilon)$ as in these theorems. Since $F[y(x; \varepsilon)]$ is identically zero, we have

$$0 = \frac{d}{d\varepsilon} F[y(x; \varepsilon)]|_{\varepsilon=0} = F'[-6/x]\left(\frac{dy}{d\varepsilon}(x; 0)\right)$$

where F' denotes the linearization of F. We conclude that this linearized equation must admit the three solutions $1/x^m$, $m = 2, 3$ and 4. If these three functions do not solve the linearization, there can be no such group action. □

The specific coefficients of the group action are not essential to the result: only the existence of an expansion of families of solutions matters.

Proof of theorem 6. The solution $y(x)$ in the statement of the theorem is constructed in theorem 8, later.

From y, we construct the one-parameter family:

$$y_2(x; \varepsilon) = -\frac{6}{x} - \frac{1}{x} h\left(\frac{\varepsilon}{x^k}\right)$$

using (5) for the inversion $x \mapsto \varepsilon^{1/k}/x$.

Letting $\varepsilon \to 0$, we find that $-(6 + h(0))/x = -(k+6)/2x$ must be a solution. We now define

$$y_1(x; \varepsilon) = -\frac{k+6}{x - \varepsilon}.$$

The properties listed in the theorem are now readily verified.   □

4. Instability of isolated poles

4.1. Results

Even though the construction of the perturbation expansion of solutions close to $-6/x$ can be made solely on the basis of the group action on solutions, the singularities which arise by perturbation of simple poles are different for (1) and (2). We know that perturbation series near a single pole do not allow an analytical description of confluence phenomena. However, even though a function such as $(x - a)^{-1} + (x - b)^{-1}$ is not jointly analytic in x, a and b small, it is the logarithmic derivative of $(x - a)(x - b)$ which is perfectly well behaved. More generally, we show that a Cole–Hopf transformation provides an analytical description of confluence phenomena in the Chazy equation.

More precisely, we have the following.

Theorem 7. For any constant a, equation (1) has precisely one solution of the form $y(x) = u'/2u$ with

$$u(x) = e^x(1 + e^x w(e^x))$$

where w is analytic when its argument is small, and $w(0) = a$. Using transformations (5), this solution generates a one-parameter family of perturbations of $-6/x$, with a natural boundary shrinking to a point as the parameter vanishes. Their asymptotics at the boundary are those suggested by the method of exponential corrections.

For equation (2), we have the following.

Theorem 8. Let a be a constant. For $k \neq 0$ or 1, equation (2) has a unique solution of the form

$$y(x) = x^{-1} h(x^k)$$

where h is analytic when its argument is small, $h(0) = (k - 6)/2$, and $h'(0) = a$. If $k = 2$, 3, 4 or 5, this solution is rational. Using transformations (5), this solution generates a one-parameter family of perturbations of $-6/x$, where all poles, except possibly one, cluster at the origin as the parameter vanishes.

Thus, $-6/x$ is unstable; the next result shows that $(k-6)/2x$ is stable, but $-(k+6)/2x$ is unstable.

Theorem 9. For $k = 2, 3, 4$ and 5, there is a three-parameter family of solutions of (2) which contains the solution $(k-6)/2x$. These parameters are in correspondence with the Cauchy data at a nearby regular point. Solutions with leading term $-(k+6)/2x$, on the other hand, are unstable under perturbation: they arise from the confluence of all singularities save one.

4.2. Remarks

(1) Since equation (14) below admits the discriminant modular form Δ as a special solution [30], theorem 7 provides a proof that Δ is entirely determined by the first two terms of its expansion in powers of $q = \exp(2\pi i x)$. Its reduction to Fuchsian form implies, in particular, a (perhaps new) recurrence relation on the coefficients of Δ, that is, on the Ramanujan τ function. On the other hand, our proof does not make use of any properties of modular forms, and therefore suggests that some of the phenomena found in the Chazy equation are of wider significance.

(2) The existence of rational solutions and the observation that a Cole–Hopf transformation simplifies some of the issues is found in Chazy [12].

4.3. Proofs

For clarity, some tedious but straightforward computations have been omitted; we sometimes found it convenient to perform some of the verifications using a symbolic manipulation package.

Proof of theorem 7. Let y be a solution of (2). Let

$$y = \frac{u'}{2u}.$$

We find that u satisfies

$$u^3 u^{(4)} - 5u^2 u' u^{(3)} - \tfrac{3}{2} u^2 u''^2 + 12uu'^2 u'' - \tfrac{13}{2} u'^4 = 0. \tag{14}$$

If u is a solution, so is

$$(cx + d)^{-12} u \left(\frac{ax + b}{cx + d} \right).$$

Since e^x is an exact solution of this equation, we seek solutions of the form $e^x v(e^x)$. Note that $u = \exp(2bx)$ leads to $y = b$, that is, to constant solutions of (1).

We make the change of variables $z = e^x$, and let $D := z \, d/dz = d/dx$. This turns (14) into a Fuchsian equation for $v(z) = u(z)/z$:

$$v^3 (D+1)^4 v - 5v^2 (D+1)v(D+1)^3 v - \tfrac{3}{2} v^2 (D+1)^2 v$$
$$+ 12v[(D+1)v]^2 (D+1)^2 v - \tfrac{13}{2}[(D+1)v]^4 = 0.$$

Letting $v(z) = 1 + zw(z)$, we find that w satisfies an equation of the form

$$(D+1)^3 Dw = zG[z, w, Dw, D^2 w, D^3 w].$$

It follows that there is exactly one solution with $w(0) = a$, and that it is given by a convergent series in z near $z = 0$.

Coming back to x, we have obtained a solution of the desired form, given by a series of exponentials which converges at least for $\operatorname{Re} x < -\rho$ for some finite ρ. We know in

fact that ρ is equal to zero for the case $u = \Delta(ix)$. This completes the proof of the claims regarding the family of exponential solutions.

Next, let us take the case when the solution y has the real axis for a natural boundary, to fix ideas.

Consider the transformation (5) generated by $x \mapsto \varepsilon x/(x - i\varepsilon)$, which maps the real axis to the circle (Γ_ε) of centre $\varepsilon/2$ and radius ε. The solution $y(x)$ generates the one-parameter family of solutions:

$$y(x; \varepsilon) = -\frac{6}{x - i\varepsilon} - \frac{i\varepsilon^2}{(x - i\varepsilon)^2} y\left(\frac{\varepsilon x}{x - i\varepsilon}\right)$$

which are defined *outside* (Γ_ε). As $\varepsilon \to 0$, we see that the natural boundary shrinks to a point, and that the solutions $y(x; \varepsilon)$ converge, uniformly on any disk at positive distance from the origin, to the solution $-6/x$.

However, the limits $\varepsilon \to 0$ and $x \to 0$ do not commute; in fact, $y(x; \varepsilon)$ is not defined in a full neighbourhood of $x = 0$ for all small values of ε.

This completes the proof of theorem 7. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of theorem 8. Let y be a solution of (2). Let

$$y = \frac{k - 6}{2} \frac{u'}{u}.$$

We find that u satisfies

$$uu^{(4)} - (k - 2)u'u''' + \frac{3k(k - 2)}{2(k + 6)}u''^2 = 0. \tag{15}$$

If u is a solution, so is

$$(cx + d)^{12/(6-k)}u\left(\frac{ax + b}{cx + d}\right).$$

The first part of the theorem follows from general results on nonlinear Fuchsian equations. Let us seek y in the form

$$y(x) = x^{-1}(a + bz + w(z)z^2)$$

where $z = x^k$, $a = (k - 6)/2$ and b is arbitrary. Letting $z\,d/dz = D$, we find, after substitution into the equation and some algebra, that w satisfies an equation of the form

$$(D + 1)(k(D + 2) + 1)(k(D + 2) - 1)w = zF[z, w, Dw, D^2w].$$

It follows that there is precisely one solution of the form $y = h(x^k)/x$ if we specify $h(0) = (k - 6)/2$ and $h'(0) = b$. This proves the first part of the theorem. If $b = 0$, we find $w \equiv 0$.

Let us now focus on $k = 2, 3, 4, 5$. In each case, there is a polynomial solution of (15), which generates the desired solutions using the SL(2) action [12]. In fact, we have

$$u = (x - a_1)\ldots(x - a_N)$$

and

$$y(x) = \frac{1}{2}(k - 6)\sum_{j=1}^{N}\frac{1}{x - a_j} = \frac{(k - 6)}{2x}\sum_{n \geqslant 0}\frac{\sum_j a_j^n}{x^n}$$

with $N = 1 + (k + 6)/(6 - k) = 12/(6 - k)$. Note that u is analytic near $x = 0$ even when the a_j tend to zero. The relation between linearized solutions and possible confluence patterns is given by the following.

Lemma 10. If we can choose the pole locations such that $a_j = \varepsilon^{1/m} b_j$, where $\sum_j b_j^q$ vanishes for $q < m$, but is non-zero if $q = m$, then

$$y(x; \varepsilon) = -\frac{6}{x} + \frac{\text{constant} \times \varepsilon}{x^{1+m}}(1 + o(1)).$$

In other words, we have a resonance at $-m$. Since the possible pole locations are obtained by applying homographic transformations to the zeros of a fixed function, not all pole configurations are possible.

The lemma follows by direct computation.

If all the poles are equal to zero, we recover $y = -6/x$; if they are all zero except for one which we let tend to infinity, we obtain the solution $-(k + 6)/x$. If all poles but one are sent to infinity, we obtain the solution $(k - 6)/x$. It is apparent that the first two solutions are unstable.

Let us now show that there cannot be any other type of confluence. Assume that there is a family of homographic transformations, depending on a parameter ε, under which two distinct poles a_1 and a_2 tend to zero while two other poles a_3 and a_4 remain fixed at non-zero (distinct) locations. We do not restrict the location of any additional poles. The anharmonic ratio of (a_1, a_2, a_3, a_4) tends to one. However, it is also independent of ε; it is therefore identically equal to one. This implies that $a_1 = a_2$ for all ε: a contradiction. Therefore, if there is such a confluence, all poles except one at most, must cluster at the same point.

Proof of theorem 9. The stability statements have already been proved in the course of the proof of the previous theorem. Since the solution $(k - 6)/2x$ has two positive resonances, namely 1 and k, we expect to be able to conclude using theorem 1. There are, in fact, no logarithms in the pole expansion, but this does not follow from theorem 7, which only generates a one-parameter solution corresponding to the resonance k: to generate the complete solution, we need to check that the resonance 1 is compatible. It is convenient to do so using the group action. More precisely, the solution $y = x^{-1}h(x^k)$ generates the solutions

$$-\frac{6\varepsilon}{1 + \varepsilon x} + \frac{h(x^k/(1 + \varepsilon x)^k)}{x(1 + \varepsilon x)}$$

which contain the additional parameter ε. Adding the translation parameter, we obtain a three-parameter family to which theorem 1 applies; this completes the proof. $\square$

Acknowledgments

References

[1] Ablowitz M J, Chakravarty S and Clarkson P A 1990 Reductions of self-dual Yang–Mills fields and classical systems *Phys. Rev. Lett.* **65** 1085–7
[2] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations, and Inverse Scattering* (Cambridge: Cambridge University Press)
[3] Ablowitz M J, Ramani A and Segur H 1980 A connection between nonlinear evolution equations and ordinary differential equations of P-type. I *J. Math. Phys.* **21** 715–21
[4] Ablowitz M J and Segur H 1977 Exact linearization of a Painlevé transcendent *Phys. Rev. Lett.* **38** 1103–6

[5] Adler M and van Moerbeke M 1982 *Inven. Math.* **76** 297–331
    Adler M and van Moerbeke M 1982 *Commun. Math. Phys.* **83** 83–106

[6] Bona J, Souganidis P and Strauss W 1987 Stability and instability of solitary waves of KdV type *Proc. R. Soc. London* A **411** 395–412

[7] Bureau F J 1972 Integration of some nonlinear systems of differential equations *Ann. Mat. Pura Appl.* (IV) **94** 345–60

[8] Bureau F J 1973 Sur des systèmes non linéaires du troisième ordre et les équations différentielles associées *Bull. Cl. Sci., Acad. Roy. Belgique* **73** 335–53

[9] Chakravarty S and Ablowitz M J 1992 On reductions of self-dual Yang–Mills equations *Painlevé Transcendents* ed D Levi and P Winternitz (New York: Plenum) pp 331–43

[10] Chazy J 1909 Sur les équations différentielles dont l'intégrale génerale est uniforme et admet des singularités essentielles mobiles *C. R. Acad. Sci. Paris* **149** 563–5

[11] Chazy J 1910 Sur les équations différentielles dont l'intégrale génerale possède une coupure essentielle mobile *C. R. Acad. Sci. Paris* **150** 456–8

[12] Chazy J 1911 Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes *Acta Math.* **34** 317–85
    Conte R 1989 Annotated table of contents, unpublished

[13] Clarkson P A and Olver P J 1996 Symmetry and the Chazy equation *J. Diff. Eq.* **124** 225–46

[14] Conte R, Fordy A P and Pickering A 1993 A perturbative Painlevé approach to nonlinear differential equations *Physica* **69D** 33–58

[15] Ercolani N and Siggia 1986 The Painlevé property and integrability *Phys. Lett.* **119A** 112
    See also Ercolani N and Siggia 1986 *Physica* **39D** 303

[16] Flaschka H, Newell A and Tabor M 1990 Integrability *What Is Integrability?* ed V Zakharov (Berlin: Springer)

[17] Fordy A P and Pickering A 1991 Analysing negative resonances in the Painlevé test *Phys. Lett.* **160A** 347–54

[18] Joshi N and Kruskal M D 1998 A local asymptotic method of seeing the natural barrier of the solutions of the Chazy equation *Applications of Analytic and Geometric Methods to Nonlinear Partial Differential Equations* ed P A Clarkson (Dordrecht: Kluwer) pp 331–40

[19] Kichenassamy S 1996 Fuchsian equations in H^s and blow-up *J. Diff. Eq.* **125** 299–327

[20] Kichenassamy S 1998 WTC expansions and non-integrable equations *Studies Appl. Math.* to appear

[21] Kichenassamy S 1995 *Nonlinear Wave Equations (Pure and Appl. Math 194)* (New York: Marcel Dekker)

[22] Kichenassamy S 1996 The blow-up problem for exponential nonlinearities *Commun. PDE* **21** 125–62

[23] Kichenassamy S and Rendall A D 1998 Analytic description of singularities in Gowdy spacetimes *Class. Quantum Grav.* to appear

[24] Kichenassamy S and Srinivasan G K 1995 The structure of WTC expansions and applications *J. Phys. A: Math. Gen.* **28** 1977–2004

[25] Koblitz N 1993 *Introduction to Elliptic Curves and Modular Forms* 2nd edn (New York: Springer)

[26] Kruskal M D and Clarkson P A 1992 The Painlevé–Kowalewski and poly-Painlevé tests for integrability *Stud. Appl. Math.* **86** 87–165

[27] Kruskal M D, Joshi N and Halburd R 1997 Analytic and asymptotic methods for nonlinear singularity analysis, a review and extension of tests for the Painlevé property *Preprint*

[28] Newell A, Tabor M and Zeng Y B 1987 A unified approach to Painlevé expansions *Physica* **29D** 1–68

[29] Pego R and Weinstein M 1992 On asymptotic stability of solitary waves *Phys. Lett.* **162A** 263–8

[30] Rankin R A 1956 The construction of automorphic forms from the derivatives of a given form *J. Indian Math. Soc.* **20** 103–16

[31] Sidorevich M P and Lukashevich N A 1990 A nonlinear third order differential equation *Diff. Eq.* **26** 450

[32] Strauss W A 1989 *Nonlinear Wave Equations (CBMS Lecture Notes 73)* (Providence, RI: American Mathematical Society)

[33] Strauss W A 1995 Stability, instability and regularity of nonlinear waves *Nonlinear Waves* to appear

[34] Takhtajan L A 1993 A simple example of modular forms as tau-functions for integrable equations *Theor. Math. Phys.* **93** 1308–17

[35] Weiss J, Tabor M and Carnevale G 1983 The Painlevé property for partial differential equations *J. Math. Phys.* **24** 522–6